Progress on Multiloop Scattering Amplitudes

new perspectives on Feynman Integral Calculus

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Alexander von Humboldt Stiftung / Foundation

Motivation

- Identify a unique Mathematical framework for any Multi-Loop Amplitude
- Simplify the calculations in High-Energy Physics
- Discover hidden properties of Feynman Amplitudes

Path

- Amplitudes Decomposition
- Multiloop *Integrand Reduction* and Multivariate Polynomial Division
- *Integrand Reduction* and the *minimal set* of Master Integrals
- Differential Equations for Feynman Integrals: Magnus Exponential
- Conclusions

Integr**and** Reduction **(Int′nd Red)**

- Very successful for **many-leg one-loop** amplitudes   Ossola, Papadopoulos, Pittau

$$
\begin{aligned}
N(q) = & \sum_{i_0<i_1<i_2<i_3}^{m-1} \left[d(i_0 i_1 i_2 i_3) + \tilde{d}(q; i_0 i_1 i_2 i_3)\right] \prod_{i \neq i_0, i_1, i_2, i_3}^{m-1} D_i \\
& + \sum_{i_0<i_1<i_2}^{m-1} \left[c(i_0 i_1 i_2) + \tilde{c}(q; i_0 i_1 i_2)\right] \prod_{i \neq i_0, i_1, i_2}^{m-1} D_i \\
& + \sum_{i_0<i_1}^{m-1} \left[b(i_0 i_1) + \tilde{b}(q; i_0 i_1)\right] \prod_{i \neq i_0, i_1}^{m-1} D_i \\
& + \sum_{i_0}^{m-1} \left[a(i_0) + \tilde{a}(q; i_0)\right] \prod_{i \neq i_0}^{m-1} D_i \\
& + \tilde{P}(q) \prod_{i}^{m-1} D_i \, .
\end{aligned}
$$

Integr**and** Reduction **(Int'nd Red)**

- Very successful for **many-leg one-loop** amplitudes

Ossola, Papadopoulos, Pittau

Integ**ral** Identities **(IBP-id's, LI-id's,...)**

Chetyrkin, Tkachov; Laporta
Gehrmann, Remiddi

- Very successful for **many-loop** up to 4-legs amplitudes

$$\int \frac{d^D k}{i\pi^{D/2}} \frac{\partial}{\partial k^\mu} v^\mu f(k, p_i) \quad = \quad 0.$$

$$2 p_i^\mu p_j^\nu \left(\sum_n p_n^{[\mu} \frac{\partial}{\partial p_n^{\nu]}} \right) I = 0$$

Integr**and** Reduction **(Int′nd Red)**

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Can we combine their advantages?

>>> *Zhang's* talk

Integr**and** Reduction **(Int′nd Red)**

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- Very successful for **many-loop** up to 4-legs amplitudes

Can we combine their advantages?

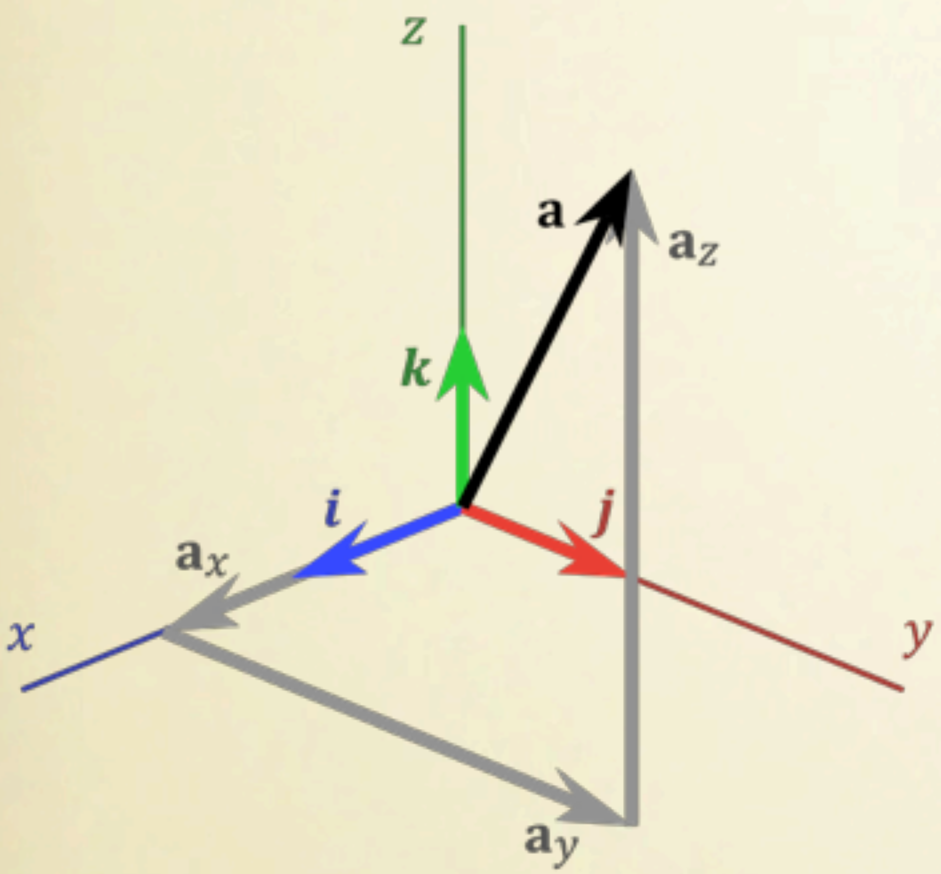
- New ideas to devise an all-order Int′nd Red′n Algorithm

Driving Principles Generic Properties of Feynman Amplitudes:

  - *Unitarity & Factorization*
  - *Loop-momentum-shift invariance*

Amplitudes Decomposition:

the algebraic way



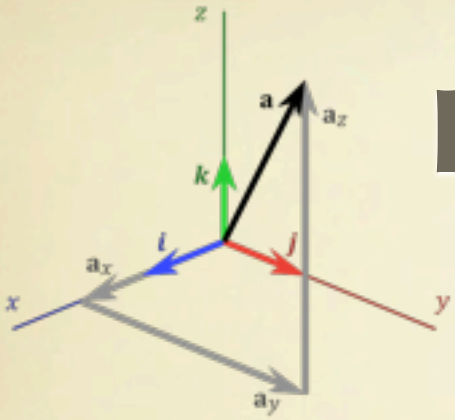
$$\mathbf{a} = a_x\,\mathbf{i} + a_y\,\mathbf{j} + a_z\,\mathbf{k}$$

- **Basis**: {**i j k**}
- **Scalar product/Projection**: to extract the components

$$a_x = \mathbf{a}.\mathbf{i}$$

$$a_y = \mathbf{a}.\mathbf{j}$$

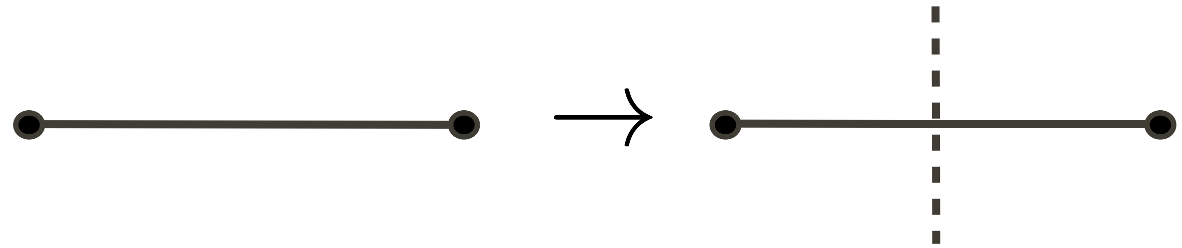
$$a_z = \mathbf{a}.\mathbf{k}$$



Projections :: On-Shell Cut-Conditions

vanishing denominators

$$\frac{1}{p^2 - m^2 - i0} \to \delta(p^2 - m^2)$$



Multi-Loop Integrand-Reduction
by
Polynomial Division

Ossola & P.M. (2011)

Badger, Frellesvig, Zhang (2011)

Zhang (2012)

Mirabella, Ossola, Peraro, & P.M (2012)

Problem: what is the form of the residues?

"find the right variables encoding the cut-structure"

variables

- ISP's = Irreducible Scalar Products:
  - q-components which can variate under cut-conditions
  - spurious: vanishing upon integration
  - non-spurious: non-vanishing upon integration $\Rightarrow$ MI's

Ossola & P.M. (2011)

A simple idea

- **Remainder Theorem**

$$\frac{f(x)}{g(x)} = q(x) + \frac{r(x)}{g(x)} \ , \qquad deg(r) < deg(g)$$

$$g(x) = (x - x_0) : \quad \Rightarrow \quad \frac{f(x)}{(x - x_0)} = q(x) + \frac{r_0}{(x - x_0)} \ , \quad r_0 = f(x_0)$$

Multivariate Polynomial Division

Zhang (2012);
Mirabella, Ossola, Peraro, & P.M. (2012)

- **Ideal** $$\mathcal{J}_{i_1\cdots i_n} = \langle D_{i_1}, \cdots, D_{i_n} \rangle \equiv \left\{ \sum_{\kappa=1}^{n} h_\kappa(\mathbf{z}) D_{i_\kappa}(\mathbf{z}) : h_\kappa(\mathbf{z}) \in P[\mathbf{z}] \right\}$$

- **Groebner Basis** $$\mathcal{G}_{i_1\cdots i_n} = \{ g_1(\mathbf{z}), \ldots, g_m(\mathbf{z}) \}$$

$$\mathcal{J}_{i_1\ldots i_n} = \langle g_1, \ldots, g_m \rangle \equiv \left\{ \sum_{\kappa=1}^{m} \tilde{h}_\kappa(\mathbf{z}) g_\kappa(\mathbf{z}) : \tilde{h}_\kappa(\mathbf{z}) \in P[\mathbf{z}] \right\}$$

n-ple cut-conditions $\quad D_{i_1} = \ldots = D_{i_n} = 0 \quad \Leftrightarrow \quad g_1 = \ldots = g_m = 0$

Multivariate Polynomial Division

Zhang (2012);
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Ideal $\mathcal{J}_{i_1 \cdots i_n} = \langle D_{i_1}, \cdots, D_{i_n} \rangle \equiv \left\{ \sum_{\kappa=1}^{n} h_\kappa(\mathbf{z}) D_{i_\kappa}(\mathbf{z}) : h_\kappa(\mathbf{z}) \in P[\mathbf{z}] \right\}$

Groebner Basis $\mathcal{G}_{i_1 \cdots i_n} = \{ g_1(\mathbf{z}), \ldots, g_m(\mathbf{z}) \}$

$$\mathcal{J}_{i_1 \ldots i_n} = \langle g_1, \ldots, g_m \rangle \equiv \left\{ \sum_{\kappa=1}^{m} \tilde{h}_\kappa(\mathbf{z}) g_\kappa(\mathbf{z}) : \tilde{h}_\kappa(\mathbf{z}) \in P[\mathbf{z}] \right\}$$

n-ple cut-conditions $D_{i_1} = \ldots = D_{i_n} = 0 \quad \Leftrightarrow \quad g_1 = \ldots = g_m = 0$

Polynomial Division $\mathcal{N}_{i_1 \cdots i_n}(\mathbf{z}) = \Gamma_{i_1 \cdots i_n} + \Delta_{i_1 \cdots i_n}(\mathbf{z}) \ ,$

Remainder ~ Residue $\Delta_{i_1 \cdots i_n}(\mathbf{z})$

Quotients $\Gamma_{i_1 \cdots i_n} = \sum_{i=1}^{m} \mathcal{Q}_i(\mathbf{z}) g_i(\mathbf{z})$ belongs to the ideal $\mathcal{J}_{i_1 \cdots i_n}$,

$$= \sum_{\kappa=1}^{n} \mathcal{N}_{i_1 \cdots i_{\kappa-1} i_{\kappa+1} \cdots i_n}(\mathbf{z}) D_{i_\kappa}(\mathbf{z}) \ .$$

Multi-Loop Integrand Recurrence

Mirabella, Ossola, Peraro, & ***P.M.*** (2012)

$$\frac{\mathcal{N}_{i_1 \ldots i_n}}{D_{i_1} \cdots D_{i_n}} = \sum_{\kappa=1}^{n} \frac{\mathcal{N}_{i_1 \ldots i_{\kappa-1} i_{\kappa+1} \ldots i_n} \; D_{i_\kappa}}{D_{i_1} \cdots D_{i_{\kappa-1}} D_{i_\kappa} D_{i_{\kappa+1}} \cdots D_{i_n}} + \frac{\Delta_{i_1 \ldots i_n}}{D_{i_1} \cdots D_{i_n}}$$

Multi-Loop Integrand Recurrence

Mirabella, Ossola, Peraro, & ***P.M.*** (2012)

$$\frac{\mathcal{N}_{i_1\ldots i_n}}{D_{i_1}\cdots D_{i_n}} = \sum_{\kappa=1}^{n} \frac{\mathcal{N}_{i_1\ldots i_{\kappa-1}i_{\kappa+1}\ldots i_n}\ \cancel{D_{i_\kappa}}}{D_{i_1}\cdots D_{i_{\kappa-1}}\cancel{D_{i_\kappa}}D_{i_{\kappa+1}}\cdots D_{i_n}} + \frac{\Delta_{i_1\ldots i_n}}{D_{i_1}\cdots D_{i_n}}$$

remainder = residue

$$\mathcal{I}_{i_1\cdots i_n} = \sum_{\kappa=1}^{k} \mathcal{I}_{i_1\cdots i_{\kappa-1}i_{\kappa+1}i_n} + \frac{\Delta_{i_1\cdots i_n}}{D_{i_1}\cdots D_{i_n}}\ .$$

n-denominator integrand

(n-1)-denominator integrand

Multi-Loop Integrand Recurrence

Mirabella, Ossola, Peraro, & ***P.M.*** (2013)

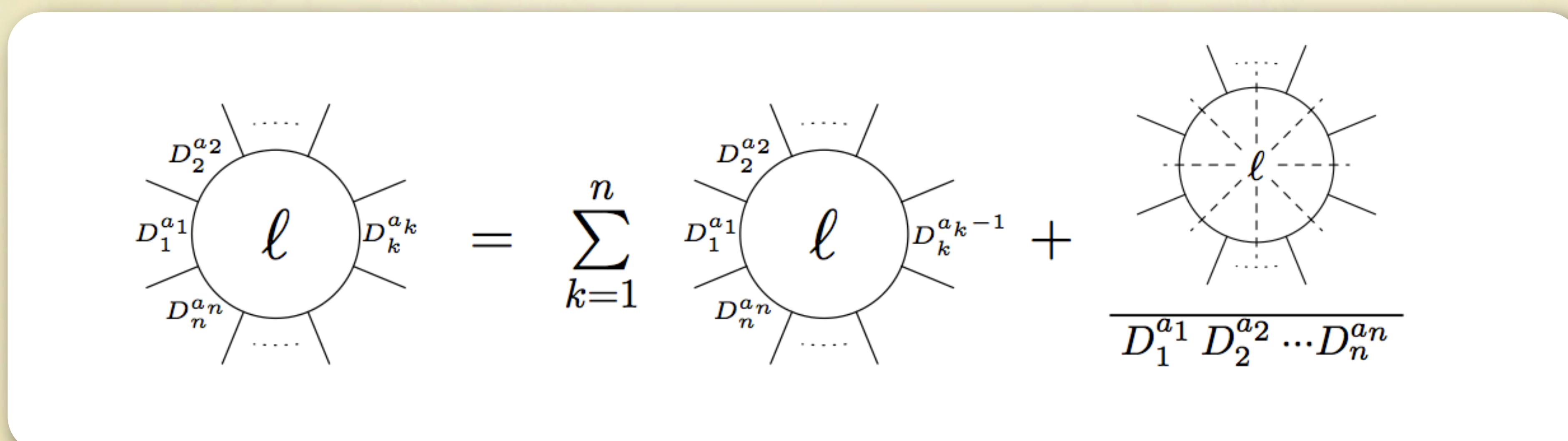
- ☑ D-reg
- ☑ Higher powers of denominators
- ☑ Arbitary kinematics

remainder = residue

$$\mathcal{I}_{\underbrace{i_1\cdots i_1}_{a_1}\cdots\underbrace{i_n\cdots i_n}_{a_n}} = \sum_{k=1}^{n} \mathcal{I}_{\underbrace{i_1\cdots i_1}_{a_1}\cdots\underbrace{i_k\cdots i_k}_{a_k-1}\cdots\underbrace{i_n\cdots i_n}_{a_n}} + \frac{\Delta_{i_1\cdots i_1\cdots i_n\cdots i_n}}{D_{i_1}^{a_1}\cdots D_{i_n}^{a_k}},$$

n-denominator integrand

(n-1)-denominator integrand



Multi-Loop Integrand Decomposition

- Multi-(particle)-pole decomposition

$$\mathcal{I}_{i_1\cdots i_n} = \frac{\mathcal{N}_{i_1\cdots i_n}}{D_{i_1} D_{i_2} \cdots D_{i_n}}$$

$$\begin{aligned}\mathcal{I}_{i_1\cdots i_n} = & \sum_{1=i_1<<i_{\max}}^{n} \frac{\Delta_{i_1 i_2 \ldots i_{\max}}}{D_{i_1} D_{i_2} \cdots D_{i_{\max}}} + \sum_{1=i_1<<i_{\max}-1}^{n} \frac{\Delta_{i_1 i_2 \ldots i_{\max}-1}}{D_{i_1} D_{i_2} \cdots D_{i_{\max}-1}} \\ & + \sum_{1=i_1<<i_{\max}-2}^{n} \frac{\Delta_{i_1 i_2 \ldots i_{\max}-2}}{D_{i_1} D_{i_2} \cdots D_{i_{\max}-2}} + \cdots\cdots + \sum_{1=i_1<i_2}^{n} \frac{\Delta_{i_1 i_2}}{D_{i_1} D_{i_2}} + \sum_{1=i_1}^{n} \frac{\Delta_{i_1}}{D_{i_1}} + Q_{\emptyset}\end{aligned}$$

Fit-on-cuts...

Knowing the parametric form of residues is ***mandatory***!!!

$$\mathcal{I}_{i_1\cdots i_n} = \sum_{1=i_1<<i_{\max}}^{n} \frac{\Delta_{i_1 i_2 \ldots i_{\max}}}{D_{i_1} D_{i_2} \cdots D_{i_{\max}}} + \sum_{1=i_1<<i_{\max}-1}^{n} \frac{\Delta_{i_1 i_2 \ldots i_{\max}-1}}{D_{i_1} D_{i_2} \cdots D_{i_{\max}-1}}$$

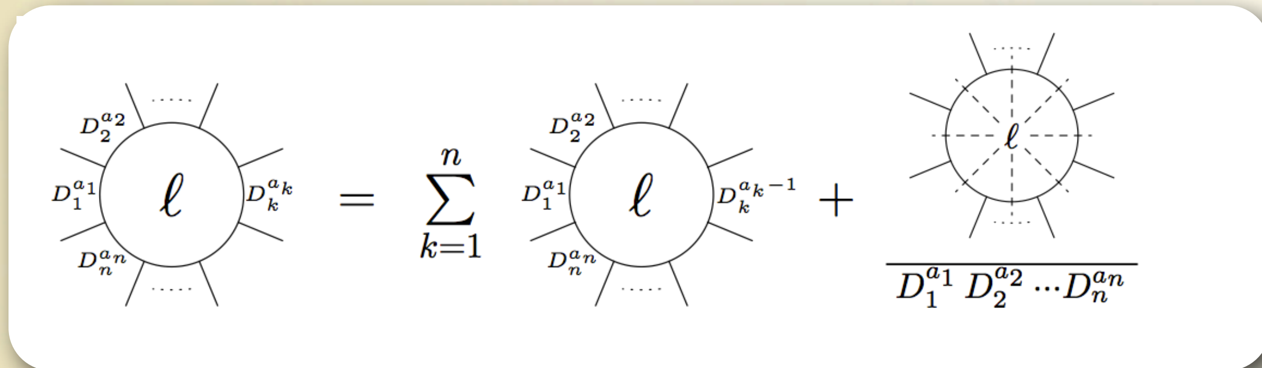
$$+ \sum_{1=i_1<<i_{\max}-2}^{n} \frac{\Delta_{i_1 i_2 \ldots i_{\max}-2}}{D_{i_1} D_{i_2} \cdots D_{i_{\max}-2}} + \cdots\cdots + \sum_{1=i_1<i_2}^{n} \frac{\Delta_{i_1 i_2}}{D_{i_1} D_{i_2}} + \sum_{1=i_1}^{n} \frac{\Delta_{i_1}}{D_{i_1}} + Q_{\emptyset}$$

*Use your favorite generator (how about **GoSam**?), and **sample** I(q's) as many time as the number of unknown coefficients*

- ☑ Parametric form of the residues is process independent.
- ☑ Actual values of the coefficients is process dependent.

...Divide and Conquer

Mirabella, Ossola, Peraro, & ***P.M.*** (2013)



remainder = residue

$$\mathcal{I}_{\underbrace{i_1\cdots i_1}_{a_1}\cdots\underbrace{i_n\cdots i_n}_{a_n}} = \sum_{k=1}^{n} \mathcal{I}_{\underbrace{i_1\cdots i_1}_{a_1}\cdots\underbrace{i_k\cdots i_k}_{a_k-1}\cdots\underbrace{i_n\cdots i_n}_{a_n}} + \frac{\Delta_{i_1\cdots i_1\cdots i_n\cdots i_n}}{D_{i_1}^{a_1}\cdots D_{i_n}^{a_k}},$$

n-denominator integrand

(n-1)-denominator integrand

just apply the ***polynomial division***
to the integrand you want to reduce:
analytic/algebraic reduction

☑ No need for the explicit cut-solutions

One-Loop Integrand-Reduction

One-Loop Integrand Decomposition

- Choice of 4-dimensional basis for an m-point residue

$$e_1^2 = e_2^2 = 0\,, \qquad e_1 \cdot e_2 = 1\,, \qquad e_3^2 = e_4^2 = \delta_{m4}\,, \qquad e_3 \cdot e_4 = -(1-\delta_{m4})$$

- Coordinates: $\mathbf{z} = (z_1, z_2, z_3, z_4, z_5) \equiv (x_1, x_2, x_3, x_4, \mu^2)$

$$q^\mu_{\text{4-dim}} = -p^\mu_{i_1} + x_1\, e_1^\mu + x_2\, e_2^\mu + x_3\, e_3^\mu + x_4\, e_4^\mu\,, \qquad q^2 = q^2_{\text{4-dim}} - \mu^2$$

- Generic numerator

$$\mathcal{N}_{i_1\cdots i_m} = \sum_{j_1,\ldots,j_5} \alpha_{\vec{j}}\, z_1^{j_1}\, z_2^{j_2}\, z_3^{j_3}\, z_4^{j_4}\, z_5^{j_5}\,, \qquad (j_1 \ldots j_5) \quad \text{such that} \quad \text{rank}(\mathcal{N}_{i_1\cdots i_m}) \le m$$

- Residues

$$\begin{aligned}
\Delta_{i_1 i_2 i_3 i_4 i_5} &= c_0 \\
\Delta_{i_1 i_2 i_3 i_4} &= c_0 + c_1 x_4 + \mu^2 (c_2 + c_3 x_4 + \mu^2 c_4) \\
\Delta_{i_1 i_2 i_3} &= c_0 + c_1 x_3 + c_2 x_3^2 + c_3 x_3^3 + c_4 x_4 + c_5 x_4^2 + c_6 x_4^3 + \mu^2 (c_7 + c_8 x_3 + c_9 x_4) \\
\Delta_{i_1 i_2} &= c_0 + c_1 x_2 + c_2 x_3 + c_3 x_4 + c_4 x_2^2 + c_5 x_3^2 + c_6 x_4^2 + c_7 x_2 x_3 + c_9 x_2 x_4 + c_9 \mu^2 \\
\Delta_{i_1} &= c_0 + c_1 x_1 + c_2 x_2 + c_3 x_3 + c_4 x_4
\end{aligned}$$

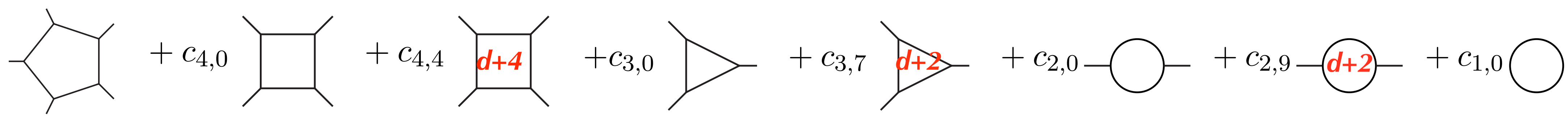
Ossola, Papadopoulos, Pittau
Ellis, Giele, Kunszt, Melnikov

One-Loop Integrand Decomposition

$$\mathcal{A}_n^{\rm one-loop} = \int d^{-2\epsilon}\mu \int d^4 q \; A_n(q,\mu^2) \; , \qquad A_n(q,\mu^2) \equiv \frac{\mathcal{N}_n(q,\mu^2)}{\bar{D}_0 \bar{D}_1 \cdots \bar{D}_{n-1}} \qquad \bar{D}_i \; = (\bar{q}+p_i)^2 - m_i^2 = (q+p_i)^2 - m_i^2 - \mu^2$$

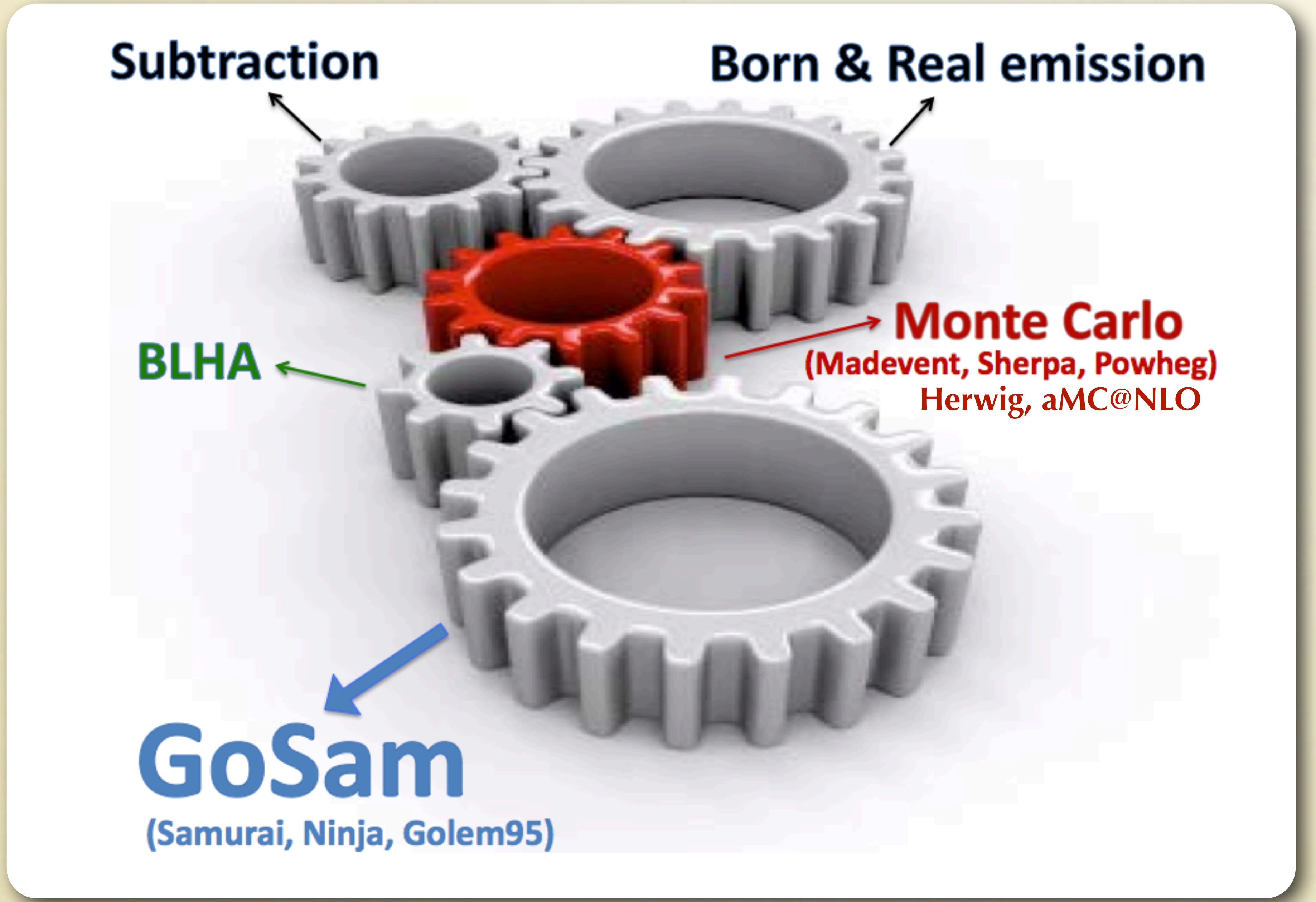
We use a bar to denote objects living in $d = \; 4 - 2\epsilon$ dimensions $\qquad \bar{\not{q}} = \not{q} + \not{\mu} \, , \quad$ with $\qquad \bar{q}^2 = q^2 - \mu^2 \; .$

$$\mathcal{A}_n^{\rm one-loop} = c_{5,0} \; + c_{4,0} \; + c_{4,4} \; + c_{3,0} \; + c_{3,7} \; + c_{2,0} \; + c_{2,9} \; + c_{1,0}$$



The GoSam Project 2.0

Cullen van Deurzen Greiner Heinrich Luisoni
Mirabella Ossola Peraro Reichel Schlenk
von Soden-Fraunhofen Tramontano ***P.M.***



MC Interfaces

Beyond SM

EW Physics

Top Physics

Diphoton and jets

Higgs (+ tops) & Jets

>>> *Heinrich's* talk

>>> *Peraro's* talk

Int'nd Red @ Higher-Loop: **it works!**

Badger, Frellesvig, Zhang

Mirabella, Ossola, Peraro, & ***P.M.***

issue:

independent monomials

are **not** a **minimal** set

***Int'nd Red @ Higher-Loop*: it works!**

Badger, Frellesvig, Zhang

Mirabella, Ossola, Peraro, & ***P.M.***

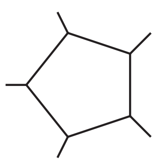
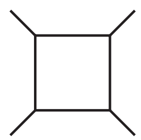
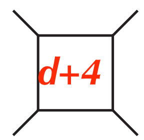
issue:
independent monomials
are **not** a **minimal** set

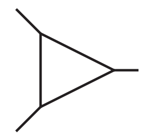
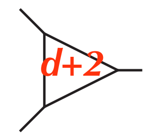
...but this is also the case at 1-loop

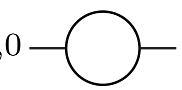
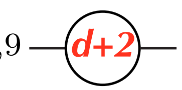
One-Loop Integrand Decomposition

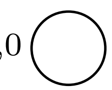
$$\mathcal{A}_n^{\mathrm{one-loop}} = \int d^{-2\epsilon}\mu \int d^4 q \; A_n(q,\mu^2)\;, \qquad A_n(q,\mu^2) \equiv \frac{\mathcal{N}_n(q,\mu^2)}{\bar{D}_0 \bar{D}_1 \cdots \bar{D}_{n-1}} \qquad \bar{D}_i = (\bar{q}+p_i)^2 - m_i^2 = (q+p_i)^2 - m_i^2 - \mu^2$$

We use a bar to denote objects living in $d = \; 4-2\epsilon$ dimensions $\qquad \bar{\not{q}} = \not{q} + \not{\mu}\,, \quad$ with $\quad \bar{q}^2 = q^2 - \mu^2\;.$

$\mathcal{A}_n^{\mathrm{one-loop}} = c_{5,0}$  $+\, c_{4,0}$  $+\, c_{4,4}$ 

$+ c_{3,0}$  $+\, c_{3,7}$ 

$+\, c_{2,0}$  $+\, c_{2,9}$ 

$+\, c_{1,0}$ 

One-Loop Integrand Decomposition

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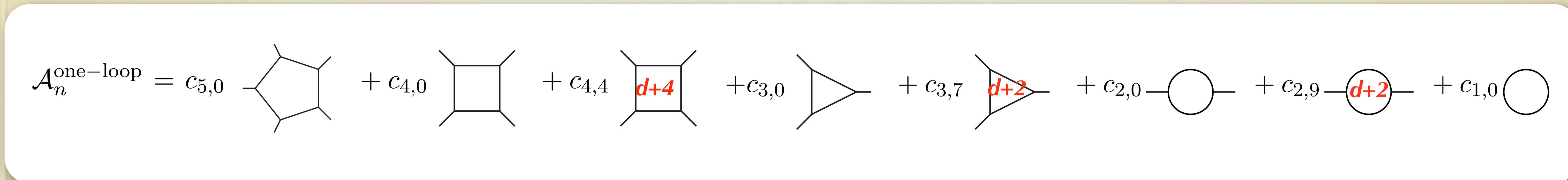
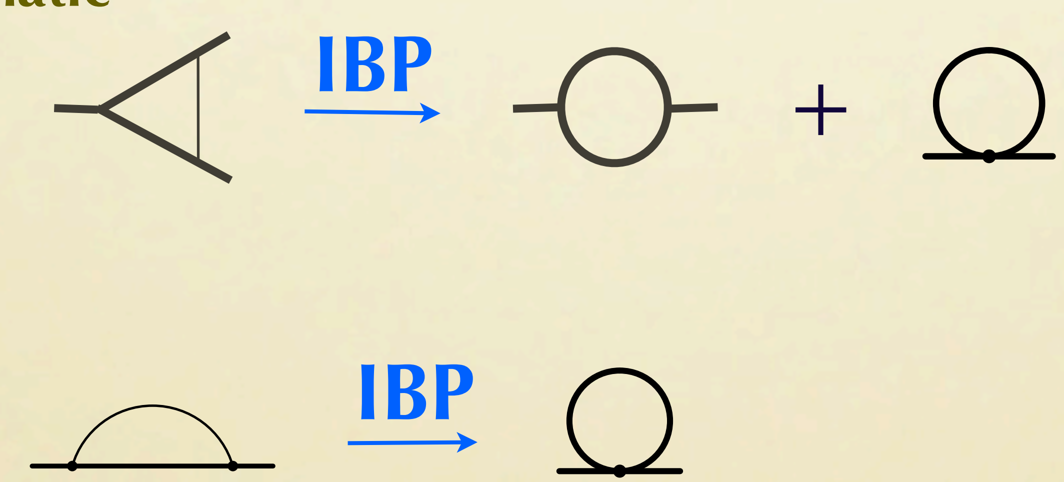
We use a bar to denote objects living in $d = 4-2\epsilon$ dimensions $\qquad \bar{\not{q}} = \not{q} + \not{\mu}\,, \quad$ with $\quad \bar{q}^2 = q^2 - \mu^2\,.$



Ex: QED-like kinematic



Solution:
Integration-by-Parts Id's
@ integrand level

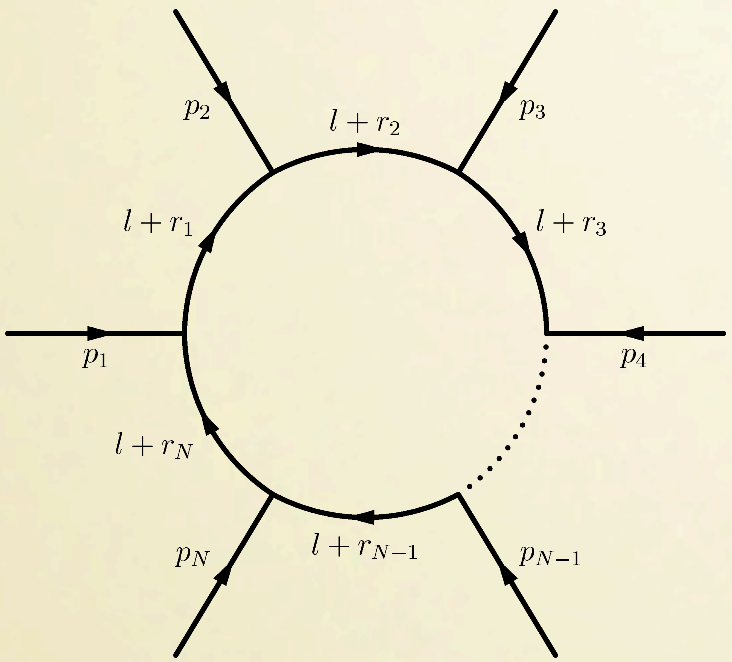
Ossola, Peraro, & ***P.M.***

Accessing the ***reducibility power*** of IBP-id's within the integrand

Let's begin with 1-Loop

1-Loop:
Dimensional-Recurrence from **IBP-id's**

Tarasov; Bern-Dixon-Kosower;
Duplancic-Nizic; Denner-Dittmaier;
Binoth-Guillet-Heinrich; ... ; Lee;



$$I_0^N(D;\{\nu_i\}) \equiv (\mu^2)^{2-D/2} \int \frac{\mathrm{d}^D l}{(2\pi)^D} \frac{1}{A_1^{\nu_1} A_2^{\nu_2} \cdots A_N^{\nu_N}} \quad .$$

$$0 \equiv \int \frac{\mathrm{d}^D l}{(2\pi)^D} \frac{\partial}{\partial l^\mu} \left(\frac{z_0 l^\mu + \sum_{i=1}^N z_i r_i^\mu}{A_1^{\nu_1} \cdots A_N^{\nu_N}} \right)$$

$$\begin{aligned} C\, I_0^N(D-2;\{\nu_k\}) \quad &= \quad \sum_{i=1}^{N} z_i I_0^N(D-2;\{\nu_k - \delta_{ki}\}) \\ &\quad +(4\pi\mu^2)(D-1-\sum_{j=1}^{N}\nu_j) z_0 I_0^N(D;\{\nu_k\}), \end{aligned}$$

Can we understand/obtain it @ integrand level?

1-Loop: Shifted-D Integrals

D = 4 - 2e

Loop Momentum Decomposition:

Mahlon; Bern-Morgan

$$\bar{q} = q + \mu \ , \qquad \bar{q}^2 = q^2 - \mu^2 \ ,$$

$\bar{q}$ in D-dimensions
q in 4-dimensions
μ in (-2ϵ)-dimensions

$$\int d^D\bar{q} \equiv \int d^{-2\epsilon}\mu \int d^4q = \int d\Omega_{-1-2\epsilon} \int_0^\infty d\mu^2 (\mu^2)^{-1-\epsilon} \ , \qquad \Omega_n \equiv \frac{2\pi^{\frac{n+1}{2}}}{\Gamma\left(\frac{n+1}{2}\right)}$$

$$I_n^D[f(q,\mu,p_i)] \equiv \int d^Dq \frac{f(q,\mu,p_i)}{D_1 \cdots D_n}$$

Dimension-raising @ Int'nd level

- From $D \to D+2$: integrand generation of $I_n^{6-2\epsilon}$:

$$I_n^{4-2\epsilon}[\mu^2] = (-\epsilon) I_n^{6-2\epsilon} \ , \qquad \frac{1}{(v_{\perp,1}\cdot v_{\perp,2})} I_n^{4-2\epsilon}[(v_{\perp,1}\cdot q)(v_{\perp,2}\cdot q)] = -\frac{1}{2} I_n^{6-2\epsilon} \qquad (v_{\perp,i}\cdot p_j = 0)$$

$$\text{(tadpole)} \quad I_1^{4-2\epsilon}[q^2] = -2 I_1^{6-2\epsilon}$$

1-Loop:
Dimensional-Recurrence from **Integrand Reduction**

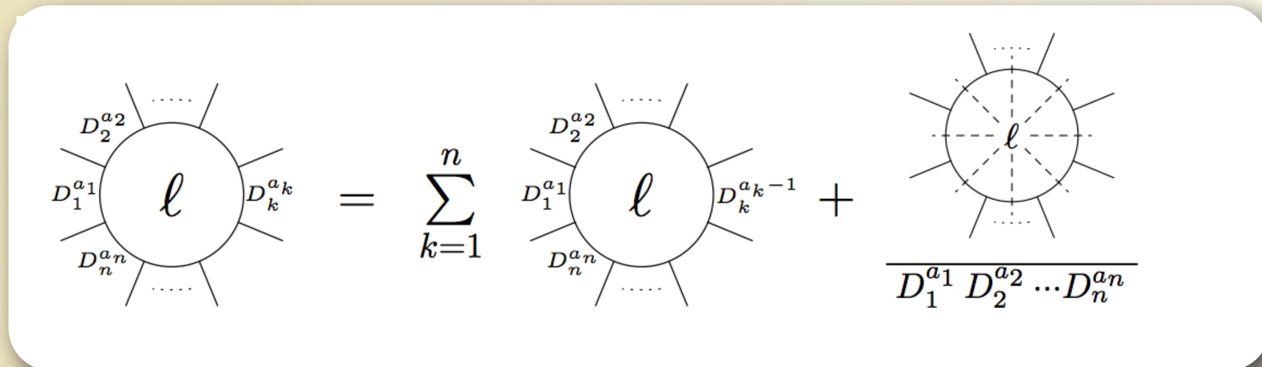
$$I_n^{D=6-2\epsilon} = \frac{1}{(n-5+2\epsilon)c_0}\left[2I_n^{D=4-2\epsilon} - \sum_{i=1}^{n} c_i I_{n-1}^{(i),D=4-2\epsilon}\right]$$

Proposition.
@ 1-Loop: Dimensional-Recurrence for I_n^D

- generated from the relation between μ^2 and $\frac{(v_{\perp,1}\cdot q)(v_{\perp,2}\cdot q)}{(v_{\perp,1}\cdot v_{\perp,2})}$ and D_i's

...Divide and Conquer

Mirabella, Ossola, Peraro, & ***P.M.*** (2013)



remainder = residue

$$\mathcal{I}_{\underbrace{i_1 \cdots i_1}_{a_1} \cdots \underbrace{i_n \cdots i_n}_{a_n}} = \sum_{k=1}^{n} \mathcal{I}_{\underbrace{i_1 \cdots i_1}_{a_1} \cdots \underbrace{i_k \cdots i_k}_{a_k - 1} \cdots \underbrace{i_n \cdots i_n}_{a_n}} + \frac{\Delta_{i_1 \cdots i_1 \cdots i_n \cdots i_n}}{D_{i_1}^{a_1} \cdots D_{i_n}^{a_k}},$$

n-denominator integrand

(n-1)-denominator integrand

just apply the ***polynomial division***
to the integrand you want to reduce:
analytic/algebraic reduction

Pentagons

We start with the 5-point one-loop integrand
$$\mathcal{I}_{01234} = \frac{\mu^2}{D_0 D_1 D_2 D_3 D_4}$$

Integrand decomposition

whose decomposition reads
$$\begin{aligned}
\mu^2 = {} & c_0^{(01234)} \\
& + \left(c_0^{(0123)} + c_1^{(0123)}(q \cdot v_\perp^{(0123)})\right) D_4 \\
& + \left(c_0^{(0124)} + c_1^{(0124)}(q \cdot v_\perp^{(0124)})\right) D_3 \\
& + \left(c_0^{(0134)} + c_1^{(0134)}(q \cdot v_\perp^{(0134)})\right) D_2 \\
& + \left(c_0^{(0234)} + c_1^{(0234)}(q \cdot v_\perp^{(0234)})\right) D_1 \\
& + \left(c_0^{(1234)} + c_1^{(1234)}((q + p_1) \cdot v_\perp^{(1234)})\right) D_0
\end{aligned}$$

Integration

$$\begin{aligned}
\mathcal{I}_{01234}[\mu^2] = -\epsilon\, \mathcal{I}_{01234}^{6-2\epsilon} = {} & c_0^{01234} \mathcal{I}_{01234} + \\
& + c_0^{(0123)}\, \mathcal{I}_{0123} + c_0^{(0124)}\, \mathcal{I}_{0124} + c_0^{(0134)}\, \mathcal{I}_{0134} \\
& + c_0^{(0234)}\, \mathcal{I}_{0234} + c_0^{(1234)}\, \mathcal{I}_{1234}.
\end{aligned}$$

Boxes

$$\mathcal{I}_{0123} = \frac{1}{v_\perp^2}\,\frac{(q\cdot v_\perp)^2}{D_0 D_1 D_2 D_3},$$

Integrand decomposition

$$\begin{aligned}\frac{(q\cdot v_\perp)^2}{v_\perp^2} = c_0^{(0123)} + \mu^2 & \\ &+ \left(c_0^{(0123)} + c_1^{(012)}(q\cdot e_3^{(012)}) + c_4^{(012)}(q\cdot e_4^{(012)})\right) D_3 \\ &+ \left(c_0^{(013)} + c_1^{(013)}(q\cdot e_3^{(013)}) + c_4^{(013)}(q\cdot e_4^{(013)})\right) D_2 \\ &+ \left(c_0^{(023)} + c_1^{(023)}(q\cdot e_3^{(023)}) + c_4^{(023)}(q\cdot e_4^{(023)})\right) D_1 \\ &+ \left(c_0^{(123)} + c_1^{(123)}(q\cdot e_3^{(123)}) + c_4^{(123)}(q\cdot e_4^{(123)})\right) D_0.\end{aligned}$$

Integration

$$\frac{1}{v_\perp^2}\mathcal{I}_n[(q.v_\perp)^2] - \mathcal{I}[\mu^2] = \frac{1}{2}(-1+2\epsilon)\,\mathcal{I}_{0123}^{6-2\epsilon} = c_0^{(0123)}\,\mathcal{I}_{0123} + \sum_{ijk} c_0^{(ijk)}\mathcal{I}_{ijk}.$$

Triangles

$$\mathcal{I}_{012} = \frac{1}{(e_3 \cdot e_4)} \frac{(q \cdot e_3)(q \cdot e_4)}{D_0 D_1 D_2},$$

Integrand decomposition

$$\frac{(q \cdot e_3)(q \cdot e_4)}{(e_3 \cdot e_4)} = c_0^{(0123)} + \frac{1}{2}\mu^2 + \text{scalar bubbles} + \text{linear bubbles} + \text{tadpoles}.$$

$$= c_0^{(0123)} + \frac{1}{2}\mu^2 + \text{scalar bubbles}.$$

Integration

$$\frac{1}{4}(-2 + 2\,\epsilon)\,\mathcal{I}_{0123}^{d=6-2\epsilon} = c_0^{(0123)} \mathcal{I}_{0123} + \sum_{ij} c_{ij} \mathcal{I}_{ij}.$$

Bubbles

$$\mathcal{I}_{01} = \frac{1}{(e_3 \cdot e_4)} \frac{(q \cdot e_3)(q \cdot e_4)}{D_0 D_1},$$

Integrand decomposition

$$\frac{(q \cdot e_3)(q \cdot e_4)}{(e_3 \cdot e_4)} = \frac{1}{2}\,\mu^2 \; + \text{scalar, linear and quadratic bubble} + \text{tadpoles.}$$

$$= \frac{1}{3}\,\mu^2 \; + \text{scalar bubble} + \text{tadpoles.}$$

Integration

$$\frac{1}{6}(-3 + 2\epsilon)\mathcal{I}_{01}^{6-2\epsilon} = c_0\;\mathcal{I}_{01} + \sum_i c_i\;\mathcal{I}_i$$

Tadpoles

Integration

$$\frac{1}{e_3 \cdot e_4} \mathcal{I}_0[(q \cdot e_3)(q \cdot e_4)] = \frac{1}{4} \mathcal{I}[\mu^2] + \frac{1}{4} m_0^2 \mathcal{I}_0$$

$$\frac{1}{8}(-4 + 2\,\epsilon)\,\mathcal{I}_0^{d=6-2\epsilon} = \frac{1}{4} m_0^2 \mathcal{I}_0.$$

or simply from $\mathcal{I}_0[q^2] = \mathcal{I}_0[\mu^2] + m_0^2\,\mathcal{I}_0$

1-Loop:

Dimensional-Recurrence: got it!

$$I_n^{D=6-2\epsilon} = \frac{1}{(n-5+2\epsilon)}\left[c_{n,0}\; I_n^{D=4-2\epsilon} - \sum_{i=1}^{n} c_{n,i}\; I_{n-1}^{(i),D=4-2\epsilon}\right]$$

$$I_{n-1}^{D=6-2\epsilon} = \frac{1}{(n-6+2\epsilon)}\left[c_{n-1,0}\; I_{n-1}^{D=4-2\epsilon} - \sum_{i=1}^{n-1} c_{n-1,i}\; I_{n-2}^{(i),D=4-2\epsilon}\right]$$

$$\ldots = \ldots$$

$$I_2^{D=6-2\epsilon} = \frac{1}{(-3+2\epsilon)}\left[c_{2,0}\; I_2^{D=4-2\epsilon} - \sum_{i=1}^{2} c_{2,i}\; I_1^{(i),D=4-2\epsilon}\right]$$

$$I_1^{D=6-2\epsilon} = \frac{1}{(-4+2\epsilon)} c_{1,0}\; I_1^{D=4-2\epsilon}$$

Dimensional Recurrence
@ integrand level:
what we can do with it?

1-Loop:
IBP-*id's* from **Dimensional-Recurrence**

$$I_n^{D=6-2\epsilon} = \frac{1}{(n-5+2\epsilon)}\left[c_{n,0}\ I_n^{D=4-2\epsilon} - \sum_{i=1}^{n} c_{n,i}\ I_{n-1}^{(i),D=4-2\epsilon}\right]$$

$$I_{n-1}^{D=6-2\epsilon} = \frac{1}{(n-6+2\epsilon)}\left[c_{n-1,0}\ I_{n-1}^{D=4-2\epsilon} - \sum_{i=1}^{n-1} c_{n-1,i}\ I_{n-2}^{(i),D=4-2\epsilon}\right]$$

$$\ldots = \ldots$$

$$I_2^{D=6-2\epsilon} = \frac{1}{(-3+2\epsilon)}\left[c_{2,0}\ I_2^{D=4-2\epsilon} - \sum_{i=1}^{2} c_{2,i}\ I_1^{(i),D=4-2\epsilon}\right]$$

$$I_1^{D=6-2\epsilon} = \frac{1}{(-4+2\epsilon)} c_{1,0}\ I_1^{D=4-2\epsilon}$$

substitute them bottom-up!

Telescopic Identity

$$(n-1+D)I_n^{D+2} = \left[c_{n,0}I_n^D - \sum_{i=1}^{n} c'_{n,i}\; I_{n-1}^{(i),D+2} - \sum_{i=1}^{n-1} c'_{n-1,i}\; I_{n-2}^{(i)D+2} - \ldots - \sum_{i=1}^{n-1} c'_{n-1,i}\; I_1^{(i),D+2}\right]$$

Sending $D \to D-2$

$$(n-3+D)I_n^{D} = \left[c_{n,0}I_n^{D-2} - \sum_{i=1}^{n} c'_{n,i}\; I_{n-1}^{(i),D} - \sum_{i=1}^{n-1} c'_{n-1,i}\; I_{n-2}^{(i)D} - \ldots - \sum_{i=1}^{n-1} c'_{n-1,i}\; I_1^{(i),D}\right]$$

Telescopic Identity

$$(n-1+D)I_n^{D+2} = \left[c_{n,0}I_n^D - \sum_{i=1}^{n} c'_{n,i}\ I_{n-1}^{(i),D+2} - \sum_{i=1}^{n-1} c'_{n-1,i}\ I_{n-2}^{(i)D+2} - \ldots - \sum_{i=1}^{n-1} c'_{n-1,i}\ I_1^{(i),D+2}\right]$$

Sending $D \to D-2$

$$(n-3+D)I_n^{D} = \left[\cancel{c_{n,0}I_n^{D-2}} - \sum_{i=1}^{n} c'_{n,i}\ I_{n-1}^{(i),D} - \sum_{i=1}^{n-1} c'_{n-1,i}\ I_{n-2}^{(i)D} - \ldots - \sum_{i=1}^{n-1} c'_{n-1,i}\ I_1^{(i),D}\right]$$

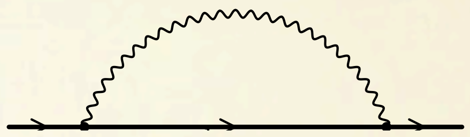
iff $c_{n,0} = 0$

$$(n-3+D)I_n^{D} = \left[- \sum_{i=1}^{n} c'_{n,i}\ I_{n-1}^{(i),D} - \sum_{i=1}^{n-1} c'_{n-1,i}\ I_{n-2}^{(i)D} - \ldots - \sum_{i=1}^{n-1} c'_{n-1,i}\ I_1^{(i),D}\right]$$

this is an IBP-id: I_n^D is reducible in terms of lower-point MI's (subtopologies).

Proposition.
$\forall n$, $c_{n,0}$ is found at the fist step of the integrand reduction, and it is not altered by the bottom-up recursive substitutions.
$\Rightarrow$ the integrand reduction can detect algebraically if I_n is MI or not.

7.1 Example: QED bubble



We consider a bubble $\mathcal{I}_{01}$ with the denominators

$$D_0 = q^2, \qquad D_1 = q^2 + 2(q \cdot p), \qquad \text{(i.e. } m_0^2 = p^2 - m_1^2 = 0).$$

- **Bubble rec. rel.** $$(1-d)\,\mathcal{I}_{01}^{(d+2)} = \mathcal{I}_1^d.$$

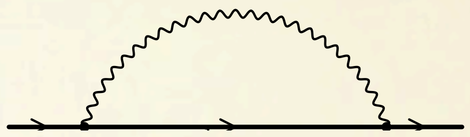
- **Tadpole rec. rel.** $$-d\,\mathcal{I}_1^{(d+2)} = 2m_e^2\,\mathcal{I}_1^{(d)}$$

- **Telescopic Identity** $$(1-d)\,\mathcal{I}_{01}^{(d+2)} = -\frac{1}{2\,m_1^2}\,d\,\mathcal{I}_1^{(d+2)},$$

shift $d \to d-2$

- **IBP-id** $$(3-d)\,\mathcal{I}_{01}^d = \frac{1}{2\,m_1^2}\,(2-d)\,\mathcal{I}_1^d.$$

7.1 Example: QED bubble



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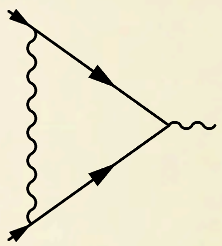
- **Telescopic Identity** $$(1-d)\,\mathcal{I}_{01}^{(d+2)} = -\frac{1}{2\,m_1^2}\,d\,\mathcal{I}_1^{(d+2)},$$

  shift $d \to d-2$
- **IBP-id** $$(3-d)\,\mathcal{I}_{01}^d = \frac{1}{2\,m_1^2}\,(2-d)\,\mathcal{I}_1^d.$$

The reduction “knows” that the integral is reducible, at its first step

7.2 Example 2 (QED vertex)

We consider a triangle $\mathcal{I}_{012}$ with kinematics corresponding to the QED vertex



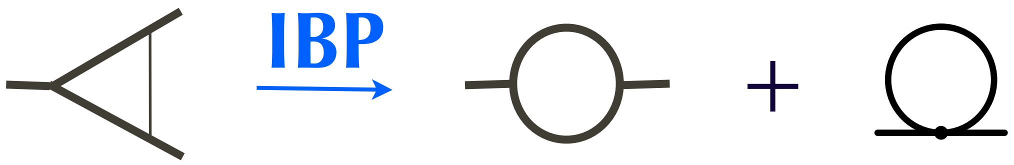
$$D_0 = \bar{q}^2, \qquad D_1 = (\bar{q}+k_1)^2 - m_e^2, \qquad D_1 = (\bar{q}-k_2)^2 - m_e^2,$$
$$\text{with} \quad m_0^2 = 0, \quad k_1^2 = k_2^2 = m_1^2 = m_2^2 = m_e^2, \quad (k_1+k_2)^2 = s.$$

- **Triangle rec. rel.** $\quad (2-d)\,\mathcal{I}_{012}^{(d+2)} = \mathcal{I}_{12}^{(d)}$
- **Bubble rec. rel.** $\quad (1-d)\,\mathcal{I}_{12}^{(d+2)} = \dfrac{4m_e^2 - s}{2}\mathcal{I}_{12}^{(d)} + \mathcal{I}_1^{(d)}$
- **Tadpole rec. rel.** $\quad -d\,\mathcal{I}_1^{(d+2)} = 2m_e^2\,\mathcal{I}_1^{(d)}$

- **Telescopic Identity** $\quad (2-d)\,\mathcal{I}_{012}^{(d+2)} = \dfrac{2}{4m_e^2 - s}\left((1-d)\,\mathcal{I}_{12}^{(d+2)} + \dfrac{d}{2\,m_e^2}\,\mathcal{I}_1^{(d+2)}\right),$

shift $d \to d-2$

- **IBP-id** $\quad (4-d)\,\mathcal{I}_{012}^{(d)} = \dfrac{2}{4m_e^2 - s}\left((3-d)\,\mathcal{I}_{12}^{(d)} + \dfrac{d-2}{2\,m_e^2}\,\mathcal{I}_1^{(d)}\right).$



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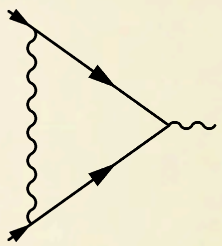
We consider a triangle $\mathcal{I}_{012}$ with kinematics corresponding to the QED vertex



$$D_0 = \bar{q}^2, \qquad D_1 = (\bar{q}+k_1)^2 - m_e^2, \qquad D_1 = (\bar{q}-k_2)^2 - m_e^2,$$
$$\text{with} \quad m_0^2 = 0, \quad k_1^2 = k_2^2 = m_1^2 = m_2^2 = m_e^2, \quad (k_1+k_2)^2 = s.$$

- **Triangle rec. rel.** $\boxed{(2-d)\,\mathcal{I}_{012}^{(d+2)} = \mathcal{I}_{12}^{(d)}}$

- **Bubble rec. rel.** $(1-d)\,\mathcal{I}_{12}^{(d+2)} = \dfrac{4m_e^2 - s}{2}\mathcal{I}_{12}^{(d)} + \mathcal{I}_{1}^{(d)}$

- **Tadpole rec. rel.** $-d\,\mathcal{I}_{1}^{(d+2)} = 2m_e^2\,\mathcal{I}_{1}^{(d)}$

- **Telescopic Identity** $(2-d)\,\mathcal{I}_{012}^{(d+2)} = \dfrac{2}{4m_e^2 - s}\left((1-d)\,\mathcal{I}_{12}^{(d+2)} + \dfrac{d}{2\,m_e^2}\,\mathcal{I}_{1}^{(d+2)}\right),$

shift $d \to d-2$

- **IBP-id** $(4-d)\,\mathcal{I}_{012}^{(d)} = \dfrac{2}{4m_e^2 - s}\left((3-d)\,\mathcal{I}_{12}^{(d)} + \dfrac{d-2}{2\,m_e^2}\,\mathcal{I}_{1}^{(d)}\right).$

The reduction "knows" that the integral is reducible, at its first step

Integrand Reduction@Shift-invariant monomials = Dimensional Recurrence ~ *IBP-id's*

mechanism

- From $D \to D+2$: integrand generation of $I_n^{6-2\epsilon}$:

$$I_n^{4-2\epsilon}[\mu^2] = (-\epsilon) I_n^{6-2\epsilon} \,, \qquad \frac{1}{(v_{\perp,1} \cdot v_{\perp,2})} I_n^{4-2\epsilon}[(v_{\perp,1} \cdot q)(v_{\perp,2} \cdot q)] = -\frac{1}{2} I_n^{6-2\epsilon}$$

$$\text{(tadpole)} \quad I_1^{4-2\epsilon}[q^2] = -2 I_1^{6-2\epsilon}$$

reducibility power of IBP-id's within the integrand: accessed!

How about 2-Loop, 3-Loop,...

Finding out the integrands that control the dimension-shift...
...better if they are also loop-momentum shift invariant

Multi-Loop:
IBP-id's from Dimensional-Recurrence

Ossola, Peraro, & ***P.M.***

- **Schwinger Parametrization**

$$\frac{1}{(p_i^2)^{\nu_i}} = \frac{1}{\Gamma(\nu_i)} \int_0^\infty dt_i \, t_i^{\nu_i - 1} \exp(-t_i p_i^2),$$

$$I^{D=4-2\epsilon}[1] = \frac{1}{(4\pi)^D} \prod_{i=1}^{7} \int_0^\infty dt_i \; \Delta^{-\frac{D}{2}} \; e^{-Q/\Delta}$$

- **Gram Determinant as Gaussian Integrals**

$$\int \left(\prod_{i=1}^{l} \frac{d^{-2\epsilon} \vec{\mu}_i}{\pi^{-\epsilon}} \right) \exp \left(\sum_{i,j=1}^{l} A_{ij} \mu_{ij} \right) = \Delta^{\epsilon}.$$

$$\mu_{ij} \leftrightarrow \frac{\partial}{\partial A_{ij}}$$

Bern, De Freitas, Dixon

Weinzierl

....

Bern, Dennen, Davies, Huang

Badger, Frellesvig, Zhang

- **D-shift Operator (D --> D+2)**

$$\frac{\Delta^{-\frac{D}{2}}}{\Delta} = \Delta^{-\frac{D+2}{2}}$$

1-Loop

$$\Delta = -\det(A_{11}) = -A_{11} \qquad \Delta^{\epsilon} = \int \exp\left(\sum_{ij} A_{ij}\mu_{ij}\right) = \int \exp(A_{11}\mu_{11}), \qquad \frac{\partial}{\partial A_{11}}\Delta^{\epsilon} = -\epsilon\,\Delta^{\epsilon} = \int \mu_{11}\,\exp(\dots),$$

$$\boxed{\mathcal{I}[\mu_{11}] = -\epsilon\,\mathcal{I}^{(d+2)}.}$$

2-Loop

$$\Delta = (-1)^2 \det\begin{pmatrix} A_{11} & A_{12} \\ A_{12} & A_{22} \end{pmatrix} = A_{11}A_{22} - A_{12}^2 \qquad \Delta^{\epsilon} = \int \exp\left(\sum_{ij} A_{ij}\mu_{ij}\right) = \int \exp(A_{11}\mu_{11} + A_{22}\mu_{22} + 2\,A_{12}\mu_{12}).$$

$$\boxed{4\,\mathcal{I}[\mu_{11}\mu_{22} - \mu_{12}^2] = 2\epsilon(1+2\epsilon)\mathcal{I}^{(d+2)}.}$$

Badger, Frellesvig, Zhang

3-Loop

$$\Delta = (-1)^3 \det\begin{pmatrix} A_{11} & A_{12} & A_{13} \\ A_{12} & A_{22} & A_{23} \\ A_{13} & A_{23} & A_{33} \end{pmatrix} \qquad \Delta^{\epsilon} = \int \exp(A_{11}\mu_{11} + A_{22}\mu_{33} + 2A_{12}\mu_{12} + 2A_{13}\mu_{13} + A_{23}\mu_{23})$$

$$= A_{13}^2 A_{22} - 2A_{12}A_{13}A_{23} + A_{11}A_{23}^2 + A_{12}^2 A_{33} - A_{11}A_{22}A_{33}.$$

$$\boxed{8\,\mathcal{I}[\mu_{13}^2\mu_{22} - 2\mu_{12}\mu_{13}\mu_{23} + \mu_{11}\mu_{23}^2 + \mu_{12}^2\mu_{33} - \mu_{11}\mu_{22}\mu_{33}] = 4\epsilon(1+\epsilon)(1+2\epsilon)\mathcal{I}^{(d+2)}.}$$

4-Loop...

Multi-Loop Dimensional-Recurrence (Int'nd level)

Ossola, Peraro, & ***P.M.***

- **Gram-Determinants/Schouten Polynomials** Remiddi, Tancredi

$$
\begin{aligned}
S(D;a) &= a^2 \\
S(D;a,b) &= a^2 b^2 - (a\cdot d)^2 \\
S(D;a;b,c) &= a^2 b^2 c^2 - a^2 (b\cdot c)^2 - b^2 (a\cdot c)^2 - c^2 (a\cdot b)^2 + 2 (a\cdot b)^2 (b\cdot c)^2 (c\cdot a)^2 \\
\ldots &= \ldots
\end{aligned}
$$

- **(-2e)-Schouten Polynomials** *[loops dependent]*

$$
\begin{aligned}
S(-2\epsilon;\mu_1) &= \mu_{11} \\
S(-2\epsilon;\mu_1,\mu_2) &= \mu_{11}\mu_{22} - \mu_{12}^2 \\
S(-2\epsilon;\mu_1,\mu_2,\mu_3) &= \mu_{11}\mu_{22}\mu_{33} - \mu_{11}\mu_{23}^2 - \mu_{22}\mu_{13}^2 - \mu_{33}\mu_{12}^2 + 2\mu_{12}^2\mu_{13}^2\mu_{23}^2 \\
\ldots &= \ldots
\end{aligned}
$$

- **(4D)-Schouten Polynomials** *[loops & legs dependent]*

$$
\begin{aligned}
& S(4;q_1)\ , \quad S(4;q_1,p_1)\ , \quad \ldots\ , \quad S(4;q_1,p_1,\ldots,p_{n-1})\ , \\
& S(4;q_1,q_2)\ , \quad S(4;q_1,q_2,p_1)\ , \quad \ldots\ , \quad S(4;q_1,q_2,p_1\ldots,p_{n-1})\ , \\
& S(4;q_1,q_2,q_3)\ , \quad S(4;q_1,q_2,q_3,p_1)\ , \quad \ldots\ , \quad S(4;q_1,q_2,q_3,p_1\ldots,p_{n-1})\ ,
\end{aligned}
$$

Multi-Loop Dimensional-Recurrence (Int'nd level)

Ossola, Peraro, & ***P.M.***

Integrand decomposition

$$S(-2\epsilon;\ldots,\mu_i,\ldots) = a_1 S(4;\ldots,q_i,\ldots p_j,\ldots) \;+ a_0 + D_i{}'\mathrm{s} + \text{spurious}$$

Integration

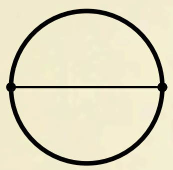
$$I_n^D[S(-2\epsilon;\ldots)] = c(\epsilon)\; I_n^{D+2}\;, \qquad I_n^D[S(4;\ldots)] = c_4\; I_n^{D+2}\;,$$

Dimensional Recurrence

$$\Big(c(\epsilon) - c_4 a_1\Big) I_n^{D+2} = a_0\; I_n^D + \text{subdiagrams}$$

Proposition.

- @ All-Loop: The Dimensional-Recurrence for I_n^D is generated from the integrand relations between $S(-2\epsilon;\mu_{ij})$, $S(4;q_{ij},p_{ij})$ and D_i's
- these relations capture the reducibility power of IBP-id's



$$\mathcal{I}_{123}[\mathcal{N}] = \frac{\mathcal{N}}{D_1 D_2 D_3}$$

$$\begin{aligned} D_1 &= \bar{q}_1^2 - m^2 = q_1^2 - m^2 - \mu_{11} \\ D_2 &= \bar{q}_2^2 - m^2 = q_2^2 - m^2 - \mu_{22} \\ D_3 &= (\bar{q}_1 - \bar{q}_2)^2 = (q_1 - q_2)^2 - \mu_{11} - \mu_{22} + 2\,\mu_{12}, \end{aligned}$$

- **Integrand decomposition** $$q_1^2 q_2^2 - (q_1 \cdot q_2)^2 = (\mu_{11}\mu_{22} - \mu_{12}^2) + m^2(\mu_1 - \mu_2)^2 + \frac{m^2}{2} D_3 + \text{spurious}$$

- **Integration** $$\mathcal{I}_{123}[q_1^2 q_2^2 - (q_1 \cdot q_2)^2] = \underline{\mathcal{I}_{123}[S(4; q_1, q_2)] = 3\, \mathcal{I}_{123}^{d+2}}$$

$$\mathcal{I}_{123}[\mu_{11}\mu_{22} - \mu_{12}^2] = \underline{\mathcal{I}_{123}[S(-2\epsilon; \mu_1, \mu_2)] = \frac{\epsilon}{2}(1 + 2\epsilon)\, \mathcal{I}_{123}^{d+2}}$$

$$\mathcal{I}_{123}[(\mu_1 - \mu_2)^2] = \; = \frac{d-4}{d} \mathcal{I}_{12}$$

- **Dimensional Recurrence**

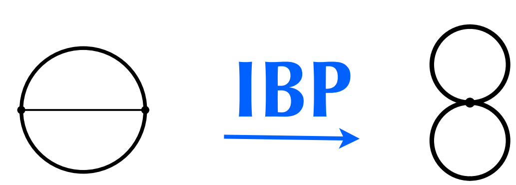
  - **2L-Vacuum** $$-\frac{1}{4}(d-1)(d-8)\, \mathcal{I}_{123}^{d+2} = \frac{4m^2}{d} \mathcal{I}_{12}^{d}$$

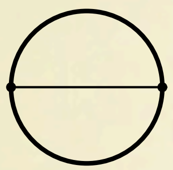
  - **(1L-Tadpole)^2** $$d^2 \mathcal{I}_{12}^{(d+2)} = 4m^4 \mathcal{I}_{12}^{(d)},$$

  - **Telescopic Id'y** $$\mathcal{I}_{123}^{(d+2)} = \frac{d}{2m^2(d-1)} \mathcal{I}_{12}^{(d+2)}.$$

$$d \to d-2$$

$$\boxed{\mathcal{I}_{123}^{d} = \frac{d-2}{2m^2(d-3)} \mathcal{I}_{12}^{d}.}$$





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- **Integrand decomposition**

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- **Integration**

$$\mathcal{I}_{123}[q_1^2 q_2^2 - (q_1 \cdot q_2)^2] = \mathcal{I}_{123}[S(4; q_1, q_2)] = 3\, \mathcal{I}_{123}^{d+2}$$

$$\mathcal{I}_{123}[\mu_{11}\mu_{22} - \mu_{12}^2] = \mathcal{I}_{123}[S(-2\epsilon; \mu_1, \mu_2)] = \frac{\epsilon}{2}(1 + 2\epsilon)\, \mathcal{I}_{123}^{d+2}$$

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- **Dimensional Recurrence**
  - **2L-Vacuum**

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  - **Telescopic Id'y**

$$\mathcal{I}_{123}^{(d+2)} = \frac{d}{2m^2(d-1)} \mathcal{I}_{12}^{(d+2)}.$$

$$d \to d-2 \qquad \mathcal{I}_{123}^{d} = \frac{d-2}{2m^2(d-3)} \mathcal{I}_{12}^{d}.$$

The reduction "knows" that the integral is reducible, at its first step

D-Shifting Operator

Tarasov; Lee;
Ossola, Peraro, Remiddi, Schubert, Tancredi, & ***P.M***.

L-loops, m-legs, n-denominators, q_i's loop momenta, p_i's external momenta;
$\vec{q} \equiv \{q_1, \ldots, q_L\}$, $\vec{p} \equiv \{p_1, \ldots, p_{m-1}\}$, $\vec{a} \equiv \{a_1, \ldots, a_n\}$

$$I^D_{m,n}[f(q_i;p_i);\vec{a}] \equiv \int d^D q_1 \cdots d^D q_L \frac{f(q_i;p_i)}{D_1^{a_1} \cdots D_n^{a_n}}$$

$$\begin{aligned} I^D_{m,n}[S(D;\vec{q},\vec{p}) f(q_i;p_i);\vec{a}] &\equiv \int d^D q_1 \cdots d^D q_L \frac{S(D;\vec{q},\vec{p}) \; f(q_i;p_i)}{D_1^{a_1} \cdots D_n^{a_n}} \\ &= \text{coeff} \times I^{D+2}_{m,n}[f(q_i;p_i);\vec{a}] \end{aligned}$$

Hence $S(D;\vec{q},\vec{p})$ plays the role of the $\mathbf{D}^+$ operator, raising $D \to D+2$.

- ☑ Easy to implement: just a polynomial in terms of of q's and p's (Gram Determinant)
- ☑ *S* is shift-invariant under redefinition of loop momentum (preserving mom. cons.)

Geometry behind Master Integrals

Ossola, Peraro, Schubert & ***P.M***.

- (towards a) **Criterion for Master Integrals**

MI's are related to the ***constant term*** of the ***Gram Determinant*** on the ***maximal cut*** of the the considered topology (where all denominators are on-shell)

[**Absence** of] *constant term* <==> [**Not**] *Master Integral*

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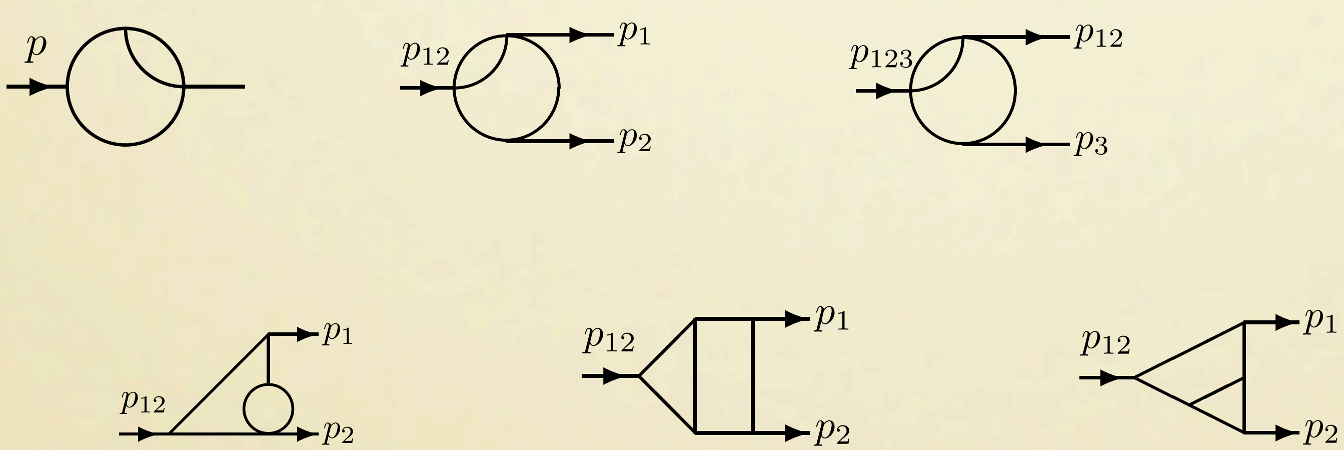
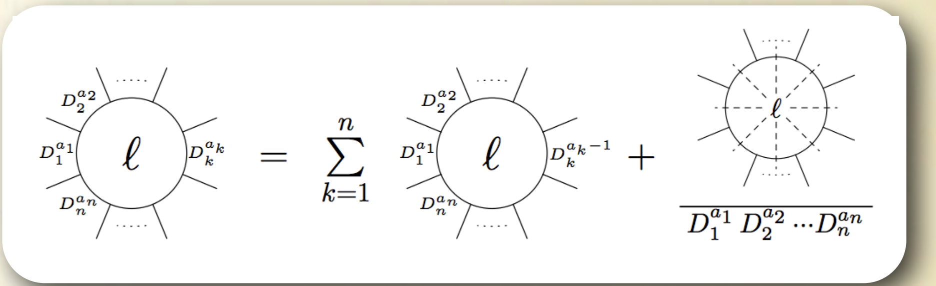
- **Examples of Reducible Integrals**



Conclusions



- **a new tool for the Decomposition of Scattering Amplitudes**
  - Multivariate Polynomial Division
    - one ingredient: Feynman denominator
    - one operation: *partial fractioning*
  - Dimensional Recurrence at the integrand level
  - embedding: Unitarity, Factorization, and loop-momentum shift invariance
  - Minimal set of MI's
- **key ideas**
  - D-shifted Master Integrals
  - Schouten Polynomials/Gram-determinants in 4- and (-2e)-dimensions
- **results**
  - reducibility criterion: purely algebraic procedure to detect MI's
  - A new, simple operator for Dimension-raising: Schouten Polynomials
- **geometry beneath**
  - Algebraic Geometry and Theory of Invariants
  - Gram-determinants ~ (iper)Volumes of polyhedra (<= Amplituhedron?)

EXTRA Slides

The Maximum-Cut Theorem

Mirabella, Ossola, Peraro, & P.M. (2012)

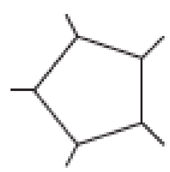
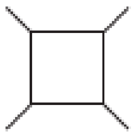
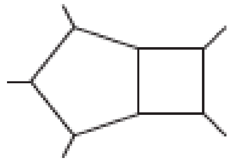
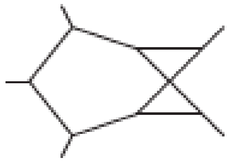
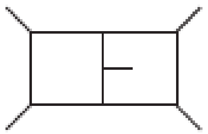
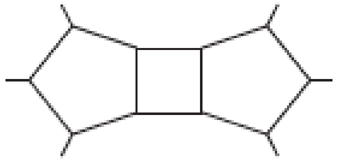
At any loop ℓ, loops we define *maximum cut* as the set of vanishing denominators

$$D_0 = D_1 = \ldots = 0$$

which constrains completely the components of the loop momenta.
We assume that, in non-exceptional phase-space points, a maximum-cut has a finite number n_s of solutions, each with multiplicity one.
Then,

Theorem 4.1 (Maximum cut). *The residue at the maximum-cut is a polynomial paramatrised by n_s coefficients, which admits a univariate representation of degree $(n_s - 1)$.*

Examples of Maximum-Cuts

diagram	Δ	n_s	diagram	Δ	n_s
	c_0	1		$c_0 + c_1 z$	2
	$\sum_{i=0}^{3} c_i z^i$	4		$\sum_{i=0}^{3} c_i z^{\,i}$	4
	$\sum_{i=0}^{7} c_i z^{\,i}$	8		$\sum_{i=0}^{7} c_i z^{\,i}$	8

- Residues

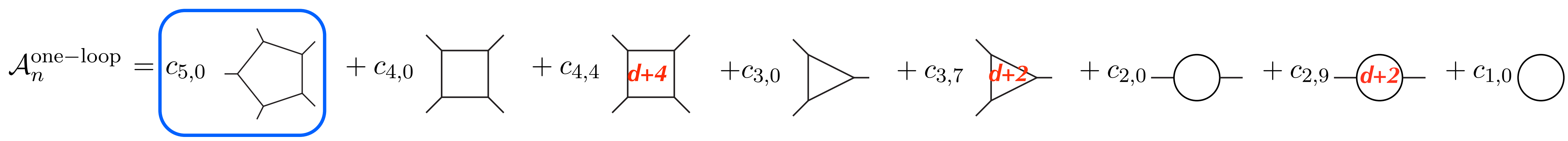
$$\boxed{\Delta_{i_1 i_2 i_3 i_4 i_5} = c_0}$$

$$\Delta_{i_1 i_2 i_3 i_4} = c_0 + c_1 x_4 + \mu^2 (c_2 + c_3 x_4 + \mu^2 c_4)$$

$$\Delta_{i_1 i_2 i_3} = c_0 + c_1 x_3 + c_2 x_3^2 + c_3 x_3^3 + c_4 x_4 + c_5 x_4^2 + c_6 x_4^3 + \mu^2 (c_7 + c_8 x_3 + c_9 x_4)$$

$$\Delta_{i_1 i_2} = c_0 + c_1 x_2 + c_2 x_3 + c_3 x_4 + c_4 x_2^2 + c_5 x_3^2 + c_6 x_4^2 + c_7 x_2 x_3 + c_9 x_2 x_4 + c_9 \mu^2$$

$$\Delta_{i_1} = c_0 + c_1 x_1 + c_2 x_2 + c_3 x_3 + c_4 x_4$$



- Residues

Samurai

Ossola, Reiter, Tramontano, & ***P.M.***

Ninja

Peraro

Mirabella, Peraro, & ***P.M.***

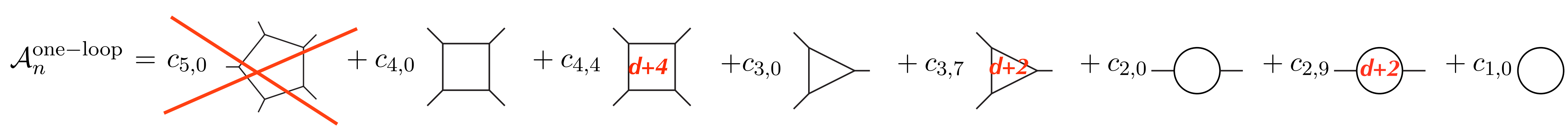
$$\boxed{\Delta_{i_1 i_2 i_3 i_4 i_5} = c_0 \, \mu^2}$$

$$\Delta_{i_1 i_2 i_3 i_4} = c_0 + c_1 x_4 + \mu^2 (c_2 + c_3 x_4 + \mu^2 c_4)$$

$$\Delta_{i_1 i_2 i_3} = c_0 + c_1 x_3 + c_2 x_3^2 + c_3 x_3^3 + c_4 x_4 + c_5 x_4^2 + c_6 x_4^3 + \mu^2 (c_7 + c_8 x_3 + c_9 x_4)$$

$$\Delta_{i_1 i_2} = c_0 + c_1 x_2 + c_2 x_3 + c_3 x_4 + c_4 x_2^2 + c_5 x_3^2 + c_6 x_4^2 + c_7 x_2 x_3 + c_9 x_2 x_4 + c_9 \mu^2$$

$$\Delta_{i_1} = c_0 + c_1 x_1 + c_2 x_2 + c_3 x_3 + c_4 x_4$$



- PV decomposition

$$I_n^{D=4-2\epsilon}[\bar{q}^\mu \bar{q}^\nu] = A_{2,0}\ \bar{g}^{\mu\nu} + \sum_{ij} A_{2,ij}\ p_i^\mu p_j^\nu$$

Contracting by $g^{\mu\nu}_{[-2\epsilon]}$:

$$I_n^{4-2\epsilon}[\mu^2] = A_{2,0}(2\epsilon) = (-\epsilon) I_n^{6-2\epsilon} \qquad \Rightarrow \qquad A_{2,0} = -\frac{1}{2} I_n^{6-2\epsilon}$$

Contracting by $v^\mu_{\perp,1} v^\nu_{\perp,2}$ with $(v_{\perp,i} \cdot p_j = 0)$:

$$I_n^{4-2\epsilon}[(v_{\perp,1} \cdot q)(v_{\perp,2} \cdot q)] = A_{2,0}(v_{\perp,1} \cdot v_{\perp,2})$$

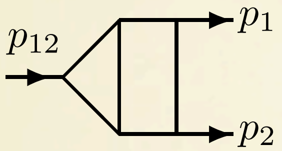
$$\Rightarrow \quad \frac{1}{(v_{\perp,1} \cdot v_{\perp,2})} I_n^{4-2\epsilon}[(v_{\perp,1} \cdot q)(v_{\perp,2} \cdot q)] = -\frac{1}{2} I_n^{6-2\epsilon}$$

- From $D \to D+2$: integrand generation of $I_n^{6-2\epsilon}$:

$$I_n^{4-2\epsilon}[\mu^2] = (-\epsilon) I_n^{6-2\epsilon}\ , \qquad \frac{1}{(v_{\perp,1} \cdot v_{\perp,2})} I_n^{4-2\epsilon}[(v_{\perp,1} \cdot q)(v_{\perp,2} \cdot q)] = -\frac{1}{2} I_n^{6-2\epsilon}$$

$$\text{(tadpole)} \quad I_1^{4-2\epsilon}[q^2] = -2 I_1^{6-2\epsilon}$$

Example of Reducible Integral




```
Schouten =
     + D6 * ( 1/8*mu12*mH^4 + 1/8*q1.e3*q2.e4*mH^4 + 1/8*q1.e4*q2.e3*mH^4 )

     + D4 * ( 1/4*q1.e3*q2.e4*q2.k1*mH^2 + 1/4*q1.e4*q2.e3*q2.k1*mH^2 + 1/4*
       q2.k1*mu12*mH^2 )

     + D3 * (  - 1/8*mu12*mH^4 - 1/4*q1.e3*q2.e4*q2.k1*mH^2 - 1/8*q1.e3*
       q2.e4*mH^4 - 1/4*q1.e4*q2.e3*q2.k1*mH^2 - 1/8*q1.e4*q2.e3*mH^4 - 1/4*
       q2.k1*mu12*mH^2 )

     + D3*D5 * ( 1/8*mu12*mH^2 + 1/8*q1.e3*q2.e4*mH^2 + 1/8*q1.e4*q2.e3*mH^2
        + q1.k1*q2.k1 + 1/2*q2.k1*mH^2 )

     + D3*D4*D5 * (  - 1/2*q2.k1 )

     + D2 * (  - 1/8*q2.k2*mH^4 )

     + D2*D6 * (  - 1/16*mH^4 )

     + D2*D5 * ( 1/16*mH^4 )

     + D2*D4 * (  - 1/8*q2.k1*mH^2 )

     + D2*D3 * (  - 3/16*mH^4 - 1/8*mu12*mH^2 - 1/8*q1.e3*q2.e4*mH^2 - 1/8*
       q1.e4*q2.e3*mH^2 - q1.k1*q2.k1 - 1/2*q1.k1*mH^2 - 3/8*q2.k1*mH^2 - 1/
       8*q2.k2*mH^2 )

     + D2*D3*D4 * ( 1/4*mH^2 + 1/2*q2.k1 )

     + D1*D5 * (  - 1/8*mu12*mH^2 - 1/8*q1.e3*q2.e4*mH^2 - 1/8*q1.e4*q2.e3*
       mH^2 )

     + D1*D4*D5 * ( 1/2*q2.k1 )

     + D1*D2 * ( 1/8*mu12*mH^2 + 1/8*q1.e3*q2.e4*mH^2 + 1/8*q1.e4*q2.e3*mH^2
        + 1/8*q2.k2*mH^2 )

     + D1*D2*D4 * (  - 1/4*mH^2 - 1/2*q2.k1 );
```